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A note on particle trajectories in the highest wave

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The series expansion procedure of Michel (1893) is used to calculate the highest wave solution and corresponding particle trajectories. The results are compared with those obtained by Longuet-Higgins (1979) using the 'hexagon' approximation for the highest deep-water wave. Reasonable agreement is found between the two sets of results.

1. Introduction

Particle trajectories in steep, symmetric, deep water waves were studied recently in a paper by Longuet-Higgins (1979). In §7 of that paper the particle trajectories and drift velocity profile for the highest wave were calculated by using the 'hexagon' approximation (originally introduced in Longuet-Higgins 1973). This note uses an accurate solution for the highest wave to calculate the particle trajectories and drift velocity. The results obtained are then compared with those of Longuet-Higgins (1979). It is found that the 'hexagon' approximation gives reasonable results for both the particle trajectories and the drift velocity profile of the highest wave.

In §2 of this note the method of calculating the highest wave solution, together with the particle trajectories and drift velocity, is given. This is followed in §3 by a comparison of the results with those obtained by the 'hexagon' approximation. In §3 the surface particle trajectory is also compared with that calculated by Longuet-Higgins (1979, §5) from Yamada's (1957) profile for the highest wave.

2. Statement of problem and method of solution

Consider steady, symmetric, periodic, gravity waves propagating on the free surface of an inviscid, incompressible fluid of infinite depth (see figure 1). Assume that the motion is two-dimensional and irrotational and that the waves are of limiting form, in which the fluid velocity at the crest is equal to the phase speed of the waves. Choose a reference frame moving with the waves, such that the origin is at the crest of one wave (see figure 1). Furthermore choose units of length and time such that the wavelength $\lambda = 2\pi$ and $g = 1$. Let ϕ and ψ denote the velocity potential and stream function, respectively, and choose $\phi = 0$ at the crest, $\psi = 0$ on the free surface and $\psi < 0$ below the free surface.

The method of solution for the highest wave on deep water, that is one having a 120° corner at the crest, is that originally proposed by Michell (1893) and recently used by Olfe & Rottman (1980) to calculate some new highest wave solutions. Here only an outline of the method is given; further details may be found in the paper by Olfe & Rottman (1980).

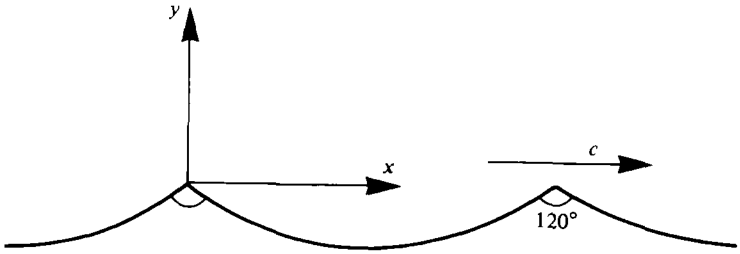

FIGURE 1. Co-ordinate system for the highest wave.

Let $\chi = \phi + i\psi$ and choose χ as the independent variable and the complex velocity $w(\chi) = u - iv = (dz/d\chi)^{-1}$ as the dependent variable (here $z = x + iy$). The function $w(\chi)$ satisfies the following conditions:

$$\frac{\partial}{\partial \phi}|w(\chi)|^4 - 4\,\mathrm{Im}\,[w(\chi)] = 0 \quad \text{on} \quad \psi = 0, \tag{1}$$

which is the ϕ derivative of Bernoulli's equation, and

$$w(\chi) \to -c \quad \text{as} \quad \psi \to -\infty, \tag{2}$$

which is the condition of uniform flow at great depth.

Assume the following expansion for $w(\chi)$

$$w(\chi) = -c[1 - \exp(-i\chi/c)]^{\frac{1}{3}} \sum_{n=0}^{\infty} a_n \exp(-in\chi/c), \tag{3}$$

which satisfies (2) if $a_0 = 1$ and also gives the 120° crest angle necessary for the limiting wave. Substitution of (3) into (1) leads to a set of nonlinear algebraic equations for the a_n and the phase speed c (for details see Olfe & Rottman 1980). The series (3) was truncated after N terms and the system of equations solved numerically for the $N+1$ unknowns $a_1, \ldots, a_N$ and c. The results obtained were checked against those of Olfe & Rottman (1980) and found to agree. For the purpose of calculating the particle trajectories, results for the case $N = 100$ are used throughout this note.

To calculate the particle trajectories in the frame of reference moving with the wave note that $\partial x/\partial \phi$, $\partial y/\partial \phi$ and $\partial y/\partial \psi$ ($= \partial x/\partial \phi$, by the Cauchy–Riemann relations) may be found from (3); hence

$$x(\phi) = \int_0^{\phi} \left.\frac{\partial x}{\partial \phi'}\right|_{\psi=\psi_c} d\phi', \tag{4}$$

$$y(\phi) = \int_0^{\psi_c} \left.\frac{\partial y}{\partial \psi}\right|_{\phi=0} d\psi + \int_0^{\phi} \left.\frac{\partial y}{\partial \phi'}\right|_{\psi=\psi_c} d\phi' \tag{5}$$

along the streamline $\psi = \psi_c (\leqslant 0)$. In (5) the first term determines the vertical distance between the crest and the highest point on the streamline $\psi = \psi_c$ (which lies below the crest), while the second term determines the variation of y, from that value, over a wavelength.

In a stationary frame of reference, past which the waves are moving, the particle trajectories may be found by adding ct to the x co-ordinate of the particle trajectories in the moving frame. Thus the particle trajectories are given by

$$X = x + ct, \quad Y = y. \tag{6}$$

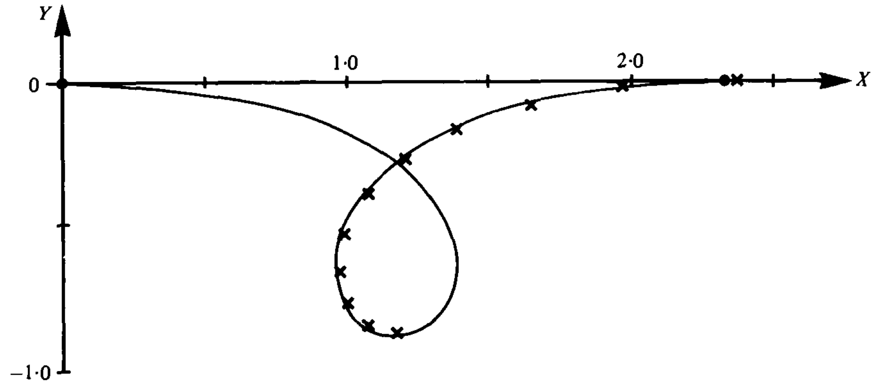

FIGURE 2. Surface particle trajectory for the highest wave. The crosses represent the particle trajectory derived by Longuet-Higgins (1979, §5) from Yamada's results.

As shown by Longuet-Higgins (1979, equation (3.3)), t is given by

$$t(\phi) = \int_0^{\phi} q^{-2}|_{\psi=\psi_c}\, d\phi, \tag{7}$$

where $q^2 = |w(\chi)|^2$, and so X and Y may easily be calculated from (4)–(7) by using (3).

Finally, if T_c is the time taken for the particle associated with the streamline $\psi = \psi_c$ to describe a complete orbit then its mean speed of advance U is given by

$$\frac{U}{c} = 1 - \frac{\lambda}{cT_c}. \tag{8}$$

The portion of a wavelength advanced by each complete orbit is given by

$$\frac{[X]}{\lambda} = \frac{cT_c}{\lambda} - 1 \tag{9}$$

(see Longuet-Higgins 1979).

The above procedure (solution of algebraic equations and evaluation of integrals) was programmed in FORTRAN and run on the Rutherford Laboratory IBM 360/195. The results obtained are presented in the next section.

3. Results and discussion

In figure 2 the surface particle trajectory is compared with that calculated by Longuet-Higgins (1979, §5) from Yamada's (1957) results for the surface profile of the highest wave. For the surface particle $U/c = 0{\cdot}273$ and $[X]/\lambda = 0{\cdot}375$, which compares well with the values 0·274 and 0·377 found from Yamada's results. From the 'hexagon' approximation $U/c = 0{\cdot}248$ and $[X]/\lambda = 0{\cdot}329$, which differ from the accurate results by approximately 10 %.

Figure 3 shows the subsurface particle trajectories and these may be compared with those given in figure 9 of Longuet-Higgins (1979). Although a direct comparison has not been made the results obtained are clearly very similar.

Finally figure 4 and table 1 give the drift velocity as a function of the mean depth $\bar{y}_c$ of the streamlines $\psi = \psi_c$ (constant). Here $\bar{y}_0$ is the mean free surface level. Also

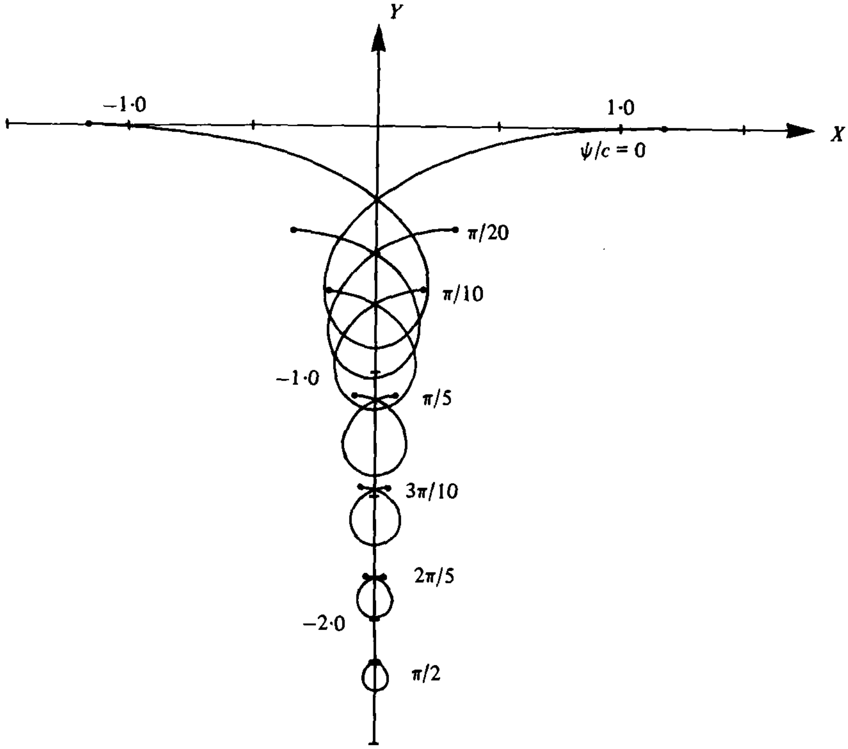

FIGURE 3. Subsurface particle trajectories for the highest waves, for various values of the parameter ψ/c.

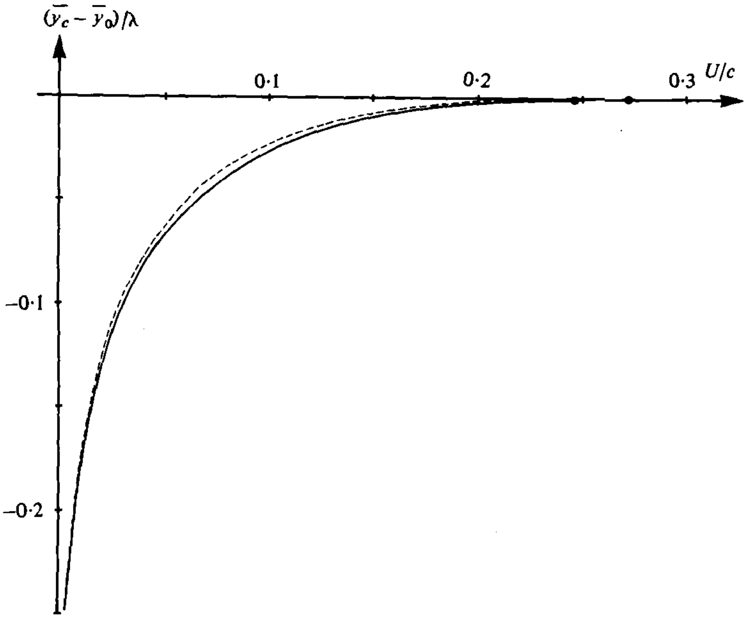

FIGURE 4. Drift velocity U/c for the highest wave as a function of the mean depth $\bar{y}_c$ of the streamlines ψ = constant. The dashed curve represents the results obtained from the 'hexagon' approximation.

$(\bar{y}_c-\bar{y}_0)/\lambda$	U/c	$(\bar{y}_c-\bar{y}_0)/\lambda$	U/c
0	0·2729	−0·0560	0·0589
−0·0021	0·1985	−0·0823	0·0395
−0·0042	0·1787	−0·1081	0·0274
−0·0062	0·1649	−0·1337	0·0193
−0·0121	0·1375	−0·1591	0·0138
−0·0178	0·1192	−0·1844	0·0099
−0·0234	0·1054	−0·2097	0·0071
−0·0290	0·0942	−0·2348	0·0052
—	—	−0·2599	0·0037

TABLE 1. Drift velocity in the highest wave as a function of the mean depth of the streamlines. (Note that: $\bar{y}_0/\lambda = -0{\cdot}0949$, $c^2 = 1{\cdot}193069$ and $h/\lambda = 0{\cdot}141067$ where h is the crest to trough height.)

shown in figure 4 are the results of Longuet-Higgins (1979) for U/c and these differ from the accurate values by about 10 %.

As can clearly be seen from the above the 'hexagon' approximation gives reasonable results when compared to those derived by an accurate method. This suggests that the 'hexagon' approximation represents the highest wave fairly accurately and can therefore be safely used as a representation of the highest wave. The accurate results given here confirm that the highest wave has a strong forward drift at the free surface, together with a strong vertical gradient of the velocity, in agreement with the results of Longuet-Higgins (1979).

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